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Research Article

A novel fractional grey model with Brute Force strategy and its applications in forecasting CO_2 emission in Brazil and China

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Abstract

One of the biggest problems facing the whole world today is climate change and the problems that arise accordingly. It is very important to estimate the emission values of CO_2 gas, which is known as the most important of the greenhouse gases causing climate change. One of the most important and reliable methods of recent years for data forecasting is grey modeling. To address this problem, a novel conformable fractional grey model, namely FEXGM(1,1), is proposed in this study. Unlike the studies in the literature, the fractional order was also used as a parameter of the model in this modeling. Thus, with the help of the optimal fraction order, the model was able to make more effective predictions. Then, estimates of the CO_2 emission value of two countries that are different from each other in terms of population and industry were made. The estimates indicate that depending on population density and industrial activities, an increase in CO_2 emissions is predicted in China, while a decrease is anticipated in Brazil. The results show that the proposed model demonstrates a higher predicted accuracy and more robust performance over the other models considered for comparison. In addition, the new grey model can be used not only for the estimation of the data in this article, but also for the estimation of all data sets.

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1. Introduction

The increase in CO_2 emissions, known as the main factor of greenhouse gas emissions, is known to have many negative effects on human health and quality of life, especially on global warming [22]. According to the BP report, it is stated that the total CO_2 emissions in the world have reached 32018.2 million tons [2]. 16.6% of total CO_2 emissions are in the North American region, 3.6% in the South American region, 11.1% in the Europe region, 6.3% in the Commonwealth of Independent States (CIS) region, 6.5% in the Middle East region, 3.9% in the Africa region and 52.4% in the Asia Pacific region. Among the countries

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in the Asia-Pacific region, China exhibits the highest level of CO_2 emissions, while Brazil occupies the top position in South America. The forecast of China and Brazil's future CO_2 emissions is very important for policymakers to formulate environmental policies to achieve the carbon emission reduction target.

One of the important subfields of grey system analysis is grey modeling. There are many methods for data estimation, the most important of which are the machine learning methods, regression analysis, hybrid models, empirical models, ARIMA, ..., etc. The most important difference of grey modeling is that it can produce low erroneous estimate with a small amount of information. In addition, it has advantageous aspects from other methods in terms of usefulness and developability. The applications of grey prediction models in various fields can be seen in the literature [4, 7, 13, 19, 25, 36, 38, 39].

It is known in mathematical modeling that the change of a quantity with respect to time is proportional to its current quantity. The mathematical expression of this idea is as follows:

$$\frac{dP(t)}{dt} = kP(t), \tag{1.1}$$

where k is proportion constant. Generally, a grey model is created by adding action terms that represent the contribution of external factors affecting the data,

$$\frac{dP(t)}{dt} = kP(t) + \text{action terms}. \tag{1.2}$$

The new grey models developed are usually obtained by replacing the action term in the whitening equation under new ideas. In this way, numerous new grey models have been developed in recent years to improve practical use and prediction accuracy [1, 3, 6, 9–12, 14, 15, 24, 27, 33].

If the data set to be estimated is regular (i.e., increasing or decreasing exponentially), first-order grey models can generally be used as a good predictor. But, if the real data set is an irregular sequence, the first-order accumulative generation operation (1-AGO) may not be implicitly procure the characteristic features of the sequence. Due to this reason, fractional derivative and fractional accumulation operators should be offered into the grey model to obviate this problem. Therefore, fractional order grey models have been developed by many researchers [5, 17, 20, 21, 23, 26, 28, 32, 35, 37]. The computational results demonstrated that the fractional models outperformed the integer-order models. Since there are explanations of the models in many studies in the literature, we did not feel the need to repeat them here [16, 30, 31, 34]. Interested readers can refer to for further details on fractional accumulation grey models.

It is the primary aim of the present work to present the novel fractional exponential grey model (abbreviated FEXGM(1,1)) by using the new definitions of conformable fractional difference and conformable fractional accumulation by Ma et al. [16]. The important difference that distinguishes this model from other its that the fractional order is also used as a parameter in the construction of the model. With our knowledge, this study is the first study in the literature in which the fractional order is also taken as a parameter. The optimal fractional order of FEXGM(1,1) is designated by using the Brute Force algorithm. Another aim is to estimate CO_2 emission data of China and Brazil. The effectiveness of FEXGM(1,1) is validated by real CO_2 emission data of China and Brazil in comparison with other used new grey models. According to the numerical outcomes, the FEXGM(1,1) model performed very well.

2. Definitions and properties of conformable fractional derivative, fractional accumulation and fractional difference

In this section, a brief overview of the conformable fractional derivative with some useful properties is given. In addition, the definitions of the conformable fractional difference and the conformable fractional accumulation by Ma et al. [16] are summarized.

2.1. The conformable fractional derivative

Definition 2.1. (see [16])) Given a differentiable function $f : [0, \infty) \to R$. Then the conformable fractional derivative of f with $\alpha \in (0, 1]$ order is defined as

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{\lceil \alpha \rceil - \alpha}) - f(t)}{\epsilon} = t^{\lceil \alpha \rceil - \alpha} \frac{df(t)}{dt} \tag{2.1}$$

where $\lceil . \rceil$ is the ceil function, i. e. the $\lceil \alpha \rceil$ is the smallest integer no larger than α. It can be clearly seen that, if $\alpha \in (0, 1]$, then $\lceil \alpha \rceil = 1$. Therefore, Eq. (2.1) can be reduced as

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon} = t^{1-\alpha} \frac{df(t)}{dt} \tag{2.2}$$

for all $t > 0$.

The properties of this new definition (2.1) are given below.

Remark 2.2. (see [8]) The α order ($\alpha \in (n, n + 1]$) conformable derivative is defined as follows

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f^{(n)}(t + \epsilon t^{\lceil \alpha \rceil - \alpha}) - f^{(n)}(t)}{\epsilon} = t^{\lceil \alpha \rceil - \alpha} \frac{d^{n+1} f(t)}{dt^{n+1}}, \tag{2.3}$$

for $\forall t > 0, \alpha \in (n, n + 1], n \in N$, and f is $(n + 1)-$ differentiable. If $\alpha \in (0, 1]$ then, Eq. (2.3) convert to Definition 1.

Theorem 2.3. *(see [8]) If the function f and g are differentiable, $\alpha \in (0, 1]$, then we have*

(1) $T_\alpha(f)(t) = t^{1-\alpha} \frac{df(t)}{dt}$
(2) $T_\alpha(rf + sg) = rT_\alpha(f) + sT_\alpha(g);\ \forall r, s \in \mathbb{R}$
(3) $T_\alpha(f.g) = fT_\alpha(g) + gT_\alpha(f)$
(4) $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$
(5) $T_\alpha(k) = 0;\ \forall k \in \mathbb{R}$
(6) $T_\alpha(t^n) = nt^{n-\alpha};\ \forall n \in \mathbb{R}$
(7) $T_\alpha(e^{kx}) = kx^{1-\alpha} e^{kx};\ \forall k \in \mathbb{R}$

Proof 1. The proof is omitted.

2.2. Fractional accumulation and difference

Recently, the new definitions to compute the conformable fractional difference and the conformable fractional accumulation are introduced by Ma et al. [16].

Definition 2.4. (See [16]) Given a differential function f, the conformable fractional difference (CFD) of f with α order is defined as

$$\Delta^\alpha f(k) = k^{1-\alpha} \Delta f(k) = k^{1-\alpha}[f(k) - f(k - 1)] \tag{2.4}$$

for all $k \in N^+$, $\alpha \in (0, 1]$ and

$$\Delta^\alpha f(k) = k^{\lceil \alpha \rceil - \alpha} \Delta^{n+1} f(k) \tag{2.5}$$

for all $k \in N^+$, $\alpha \in (n, n + 1]$.

Definition 2.5. (See [16]) The conformable fractional accumulation (CFA) of f with α order is defined as

$$\nabla^{\alpha} f(k) = \nabla\left(\frac{f(k)}{k^{1-\alpha}}\right) = \sum_{j=1}^{k} \frac{f(j)}{j^{1-\alpha}} \tag{2.6}$$

for all $k \in N^{+}$, $\alpha \in (0,1]$ and

$$\nabla^{\alpha} f(k) = \nabla^{n+1}\left(\frac{f(k)}{k^{\lceil\alpha\rceil-\alpha}}\right) \tag{2.7}$$

for all $k \in N^{+}$, $\alpha \in (n, n+1]$.

3. New fractional exponential grey model FEXGM(1,1)

A novel fractional exponential grey model (FEXGM(1,1)) is introduced, in this section. The proposed grey model has been strengthened via an exponential grey action quantity depend on α. Thus, α plays an active role not only as a fractional order, but also in determining the parameters of the model.

3.1. Formulation of proposed fractional grey model

With the original series $X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), ..., x^{(0)}(n)\right)$ we firstly denote the α order CFA as

$$X^{(\alpha)} = \left(x^{(\alpha)}(1), x^{(\alpha)}(2), ..., x^{(\alpha)}(n)\right) \tag{3.1}$$

where

$$x^{(\alpha)}(k) = \nabla^{\alpha} x^{(0)}(k) = \sum_{i=1}^{k} \begin{bmatrix}\lceil\alpha\rceil \\ k-i\end{bmatrix} \frac{x^{(0)}(i)}{i^{\lceil\alpha\rceil-\alpha}}, \alpha \in \mathbb{R}^{+} \tag{3.2}$$

where $\lceil\cdot\rceil$ is the ceil function and $\begin{bmatrix}\lceil\alpha\rceil \\ k-i\end{bmatrix} = \frac{\Gamma(k-i+\lceil\alpha\rceil)}{\Gamma(k-i+1)\Gamma(\lceil\alpha\rceil)} = \frac{(k-i+\lceil\alpha\rceil-1)!}{(k-i)!(\lceil\alpha\rceil-1)!}$ [29].

Definition 3.1. The first-order differential equation

$$\frac{dx^{(\alpha)}(t)}{dt} + ax^{(\alpha)}(t) = b + c(\alpha - \lceil\alpha\rceil)e^{(\alpha-\lceil\alpha\rceil)t} \tag{3.3}$$

is known as the whitening differential equation of the FEXGM(1, 1), in which a is a development coefficient, b is called driving coefficient and $c(\alpha - \lceil\alpha\rceil)e^{(\alpha-\lceil\alpha\rceil)t}$ is an exponential grey action quantity. When $c = 0$, it yields the CFGM(1,1) [16]. In addition, the FEXGM(1,1) model is converted to the standard GM(1,1) model for $\alpha = 1$ [36].

The monotonically decreasing exponential term in equation (3.3) was introduced to suppress prediction errors during sudden changes in the data over time. One of the reasons for choosing the form "$(\alpha - \lceil\alpha\rceil)$" is that when $\alpha = 1$, the model reduces to the standard grey model, thus encompassing it as a special case. Other motivations include the fact that the solution of the differential equation becomes analytically simple, and that the exponent in the exponential action term is constrained to be negative, thereby ensuring its monotone decrease.

3.2. Parameters estimation

Theorem 3.2. *With given data and the value of fractional order, the system parameters a, b ve c of the FEXGM(1, 1) model are*

$$[a, b, c]^{T} = \left(B^{T}B\right)^{-1} B^{T}Y \tag{3.4}$$

where the matrix B and Y are

$$B=\begin{bmatrix} -0.5(x^{(\alpha)}(2)+x^{(\alpha)}(1)) & 1 & e^{2(\alpha-\lceil\alpha\rceil)}-e^{(\alpha-\lceil\alpha\rceil)} \\ -0.5(x^{(\alpha)}(3)+x^{(\alpha)}(2)) & 1 & e^{3(\alpha-\lceil\alpha\rceil)}-e^{2(\alpha-\lceil\alpha\rceil)} \\ . & . & . \\ . & . & . \\ -0.5(x^{(\alpha)}(n)+x^{(\alpha)}(n-1)) & 1 & e^{n(\alpha-\lceil\alpha\rceil)}-e^{(n-1)(\alpha-\lceil\alpha\rceil)} \end{bmatrix}, Y=\begin{bmatrix} x^{(\alpha)}(2)-x^{(\alpha)}(1) \\ x^{(\alpha)}(3)-x^{(\alpha)}(2) \\ . \\ . \\ x^{(\alpha)}(n)-x^{(\alpha)}(n-1) \end{bmatrix}, \rho=\begin{bmatrix} a \\ b \\ c \end{bmatrix}. \quad (3.5)$$

Proof 2. Integrating left and right sides of equation (3.3) from $k-1$ to k and using the trapezoid formula for the second integral, the discrete form of FEXGM(1,1) model is found as follow:

$$\int_{k-1}^{k}\frac{dx^{(\alpha)}(t)}{dt}dt+a\int_{k-1}^{k}x^{(\alpha)}(t)dt=\int_{k-1}^{k}(b+c(\alpha-\lceil\alpha\rceil)e^{(\alpha-\lceil\alpha\rceil)t})dt, \quad (3.6)$$

so

$$\left(x^{(\alpha)}(k)-x^{(\alpha)}(k-1)\right)+\tfrac{a}{2}\left(x^{(\alpha)}(k)+x^{(\alpha)}(k-1)\right)=b+c\left(e^{(\alpha-\lceil\alpha\rceil)k}-e^{(\alpha-\lceil\alpha\rceil)(k-1)}\right) \quad (3.7)$$

where $k=2,3,...,n$.

The system (3.7) can be expanded as follows

$$\begin{aligned} x^{(\alpha)}(2)-x^{(\alpha)}(1)&=-0.5a(x^{(\alpha)}(2)+x^{(\alpha)}(1))+b+c(e^{2(\alpha-\lceil\alpha\rceil)}-e^{(\alpha-\lceil\alpha\rceil)}) \\ x^{(\alpha)}(3)-x^{(\alpha)}(2)&=-0.5a(x^{(\alpha)}(3)+x^{(\alpha)}(2))+b+c(e^{3(\alpha-\lceil\alpha\rceil)}-e^{2(\alpha-\lceil\alpha\rceil)}) \\ &\vdots \\ x^{(\alpha)}(n)-x^{(\alpha)}(n-1)&=-0.5a(x^{(\alpha)}(n)+x^{(\alpha)}(n-1))+b+c(e^{n(\alpha-\lceil\alpha\rceil)}-e^{(n-1)(\alpha-\lceil\alpha\rceil)}) \end{aligned} \quad (3.8)$$

Therefore system (3.8) can be represented as a matrix system as follows

$$Y=B\rho, \quad (3.9)$$

where

$$B=\begin{bmatrix} -0.5(x^{(\alpha)}(2)+x^{(\alpha)}(1)) & 1 & e^{2(\alpha-\lceil\alpha\rceil)}-e^{(\alpha-\lceil\alpha\rceil)} \\ -0.5(x^{(\alpha)}(3)+x^{(\alpha)}(2)) & 1 & e^{3(\alpha-\lceil\alpha\rceil)}-e^{2(\alpha-\lceil\alpha\rceil)} \\ . & . & . \\ . & . & . \\ -0.5(x^{(\alpha)}(n)+x^{(\alpha)}(n-1)) & 1 & e^{n(\alpha-\lceil\alpha\rceil)}-e^{(n-1)(\alpha-\lceil\alpha\rceil)} \end{bmatrix}, Y=\begin{bmatrix} x^{(\alpha)}(2)-x^{(\alpha)}(1) \\ x^{(\alpha)}(3)-x^{(\alpha)}(2) \\ . \\ . \\ x^{(\alpha)}(n)-x^{(\alpha)}(n-1) \end{bmatrix}, \rho=\begin{bmatrix} a \\ b \\ c \end{bmatrix}. \quad (3.10)$$

The FEXGM(1,1) model's parameter sequence ρ estimate can be accomplished by employing the least squares method as,

$$[a,b,c]^T=\left(B^TB\right)^{-1}B^TY. \quad (3.11)$$

This proof is completed.

3.3. Response function and restored values

Theorem 3.3. *The discrete form of the response function of FEXGM(1, 1) model is given as*

$$x^{(\alpha)}(k)=\tfrac{b}{a}+\tfrac{c(\alpha-\lceil\alpha\rceil)}{\alpha-\lceil\alpha\rceil+a}e^{(\alpha-\lceil\alpha\rceil)k}+\left(x^{(0)}(1)-\tfrac{b}{a}-\tfrac{c(\alpha-\lceil\alpha\rceil)}{\alpha-\lceil\alpha\rceil+a}e^{\alpha-\lceil\alpha\rceil}\right)e^{a(1-k)} \quad (3.12)$$

where $k=2,3,...,n$.

Proof 3. The solution of equation (3.3) is

$$\hat{x}^{(\alpha)}(t)=\frac{b}{a}+\frac{c(\alpha-\lceil\alpha\rceil)}{\alpha-\lceil\alpha\rceil+a}e^{(\alpha-\lceil\alpha\rceil)t}+Me^{-at} \quad (3.13)$$

where M is integral constant. The constant M can be obtained by using the initial condition $x^{(\alpha)}(1) = x^{(0)}(1)$,

$$M = \left(x^{(0)}(1) - \frac{b}{a} - \frac{c(\alpha - \lceil \alpha \rceil)}{\alpha - \lceil \alpha \rceil + a} e^{\alpha - \lceil \alpha \rceil} \right) e^{a}.$$

Accordingly, the Eq. (3.13) can be written as following

$$\hat{x}^{(\alpha)}(t) = \frac{b}{a} + \frac{c(\alpha - \lceil \alpha \rceil)}{\alpha - \lceil \alpha \rceil + a} e^{(\alpha - \lceil \alpha \rceil)t} + \left(x^{(0)}(1) - \frac{b}{a} - \frac{c(\alpha - \lceil \alpha \rceil)}{\alpha - \lceil \alpha \rceil + a} e^{\alpha - \lceil \alpha \rceil} \right) e^{a(1-t)}$$

and the response function can be written in discrete form as follows

$$\hat{x}^{(\alpha)}(k) = \frac{b}{a} + \frac{c(\alpha - \lceil \alpha \rceil)}{\alpha - \lceil \alpha \rceil + a} e^{(\alpha - \lceil \alpha \rceil)k} + \left(x^{(0)}(1) - \frac{b}{a} - \frac{c(\alpha - \lceil \alpha \rceil)}{\alpha - \lceil \alpha \rceil + a} e^{\alpha - \lceil \alpha \rceil} \right) e^{a(1-k)}.$$

The proof is completed.

Theorem 3.4. *Then restored values of FEXGM(1,1) model can be written as*

$$\hat{x}^{(0)}(k) = \Delta^{\alpha} \hat{x}^{(\alpha)}(k) = k^{\lceil \alpha \rceil - \alpha} \Delta^{n+1} \hat{x}^{(\alpha)}(k), \ \alpha \in (n, n+1] \tag{3.14}$$

where $k = 2, 3, ..., n$.

Proof 4. By using Eq. (3.2) it is clear that $\hat{x}^{(\alpha)}(k) = \nabla^{\alpha} \hat{x}^{(0)}(k)$. Using the inverse operator Δ^{α}, the restored values can be written in implicit form as

$$\hat{x}^{(0)}(k) = \Delta^{\alpha} \hat{x}^{(\alpha)}(k).$$

By using the definition 2, the restored values of FEXGM(1,1) model can be obtained as

$$\hat{x}^{(0)}(k) = \Delta^{\alpha} \hat{x}^{(\alpha)}(k) = k^{\lceil \alpha \rceil - \alpha} \Delta^{n+1} \hat{x}^{(\alpha)}(k), \ \alpha \in (n, n+1].$$

Particularly, when $\alpha \in (0, 1]$ the restored values can be written as

$$\hat{x}^{(0)}(k) = k^{1-\alpha} (\hat{x}^{(\alpha)}(k) - \hat{x}^{(\alpha)}(k-1)). \tag{3.15}$$

3.4. Evaluative accuracy of the forecating model

To evaluate the prediction accuracy of FEXGM(1, 1), relative percentage error (RPE) and the mean absolute percentage error (MAPE) are used to evaluate the level of prediction performance of the prediction models. They are defined as follows:

$$RPE(k) = \left| \frac{\hat{x}^{(0)}(k) - x^{(0)}(k)}{x^{(0)}(k)} \right| \times 100\% \tag{3.16}$$

$$MAPE = \frac{1}{n-1} \sum_{k=2}^{n} RPE(k) \tag{3.17}$$

The metrics $MAPE_{fit}$ and $MAPE_{pre}$ are defined as the fitting and prediction performance metrics for a raw sequence of n samples, respectively. These can be expressed mathematically as follows:

$$MAPE_{fit} = \frac{1}{m-1} \sum_{k=2}^{m} RPE(k) \tag{3.18}$$

$$MAPE_{pre} = \frac{1}{n-m} \sum_{k=m+1}^{n} RPE(k) \tag{3.19}$$

where m is the number of samples used to build a model. Therefore the remaining $n - m$ terms of the raw sequence is used to measure the prediction accuracy of the model. The total MAPE is given Eq. (3.17) or Eq. (3.20) and it is used to evaluate the whole performance of a model,

$$MAPE_{tot} = \frac{1}{n-1} \left((m-1) MAPE_{fit} + (n-m) MAPE_{pre} \right). \tag{3.20}$$

3.5. Optimization of the fractional order α based on Brute Force algorithm

The Brute Force method, also known as the Naive approach, is a simple and direct way to solve optimization problems by checking all possible solutions in the search space. Unlike heuristic or intelligent algorithms that try to improve efficiency by using adaptive strategies, Brute Force explores every option without relying on such techniques. A key advantage of this method is that it avoids getting stuck in local optima, which is a common issue in many popular optimization algorithms. Although those methods may converge faster, they sometimes fail to find the best overall solution. In contrast, the Brute Force method examines the entire search space, calculates the MAPE at each point, and identifies the best parameters with high reliability.

Our experiments showed that, 91% optimum values of α are in the edge of (0,1) and 9% optimum values of α are obtained at 1 in the FEXGM model. It is clear that, the FEXGM(1,1) model can be translated to the standard GM(1,1) model for $\alpha = 1$. In the FGM model, 72% optimum values of α are in the range of (0,1) and 99% optimum values of α in the CFGM model are in the range of [0,1) (for more detailed information, see [16]).

4. Mechanism and Validation of the FEXGM(1,1)

In this section, to demonstrate the computational mechanism of the FEXGM(1,1), two explicative examples are provided.

4.1. Example 1

In here, we choose the raw data as the following monotone rising series,

$$X^{(0)}(k) = (44.21, 51.57, 57.86, 61.80, 67.80, 70.31, 75.87). \tag{4.1}$$

According to Brute Force algorithm, the optimal α value was searched in the interval (0,1) with a step size of 0.01. The MAPE values calculated at each point can be seen in Table 1 and Figure 1. According to Figure 1, the optimum α is attained at $\alpha = 0.56$, and the MAPE is computed for the optimum α as MAPE= 0.6789 for the given raw data set.

4.1.1. Parameter estimation and conformable fractional accumulation. By using Eq. (3.2), the CFA series for $\alpha = 0.56$ can be written as

$$X^{(0.56)}(k) = (44.2100, 82.2752, 117.7748, 151.6814, 184.6214, 216.9987, 249.0737).$$

The matrix B and Y are constructed as

$$B = \begin{bmatrix} -63.2170 & 1 & -0.2292 \\ -100.0648 & 1 & -0.1476 \\ -134.6957 & 1 & -0.0950 \\ -168.1833 & 1 & -0.0612 \\ -200.8617 & 1 & -0.0394 \\ -232.9564 & 1 & -0.0254 \end{bmatrix}, \quad Y = \begin{bmatrix} 38.0140 \\ 35.6816 \\ 33.5801 \\ 33.3951 \\ 31.9617 \\ 32.2275 \end{bmatrix}.$$

Table 1. MAPEs of FEXGM(1, 1) model with α in $(0, 1]$ by step 0.01.

α	MAPE	α	MAPE	α	MAPE	α	MAPE	α	MAPE
0.01	0.9058	0.21	0.7646	0.41	0.6952	0.61	0.6812	0.81	0.7305
0.02	0.8966	0.22	0.7600	0.42	0.6931	0.62	0.6821	0.82	0.7346
0.03	0.8876	0.23	0.7555	0.43	0.6911	0.63	0.6832	0.83	0.7388
0.04	0.8787	0.24	0.7511	0.44	0.6893	0.64	0.6845	0.84	0.7431
0.05	0.8700	0.25	0.7468	0.45	0.6876	0.65	0.6859	0.85	0.7476
0.06	0.8615	0.26	0.7427	0.46	0.6861	0.66	0.6874	0.86	0.7522
0.07	0.8531	0.27	0.7386	0.47	0.6847	0.67	0.6892	0.87	0.7568
0.08	0.8448	0.28	0.7347	0.48	0.6835	0.68	0.6911	0.88	0.7615
0.09	0.8368	0.29	0.7309	0.49	0.6824	0.69	0.6932	0.89	0.7663
0.10	0.8289	0.30	0.7272	0.50	0.6814	0.70	0.6954	0.90	0.7711
0.11	0.8211	0.31	0.7237	0.51	0.6806	0.71	0.6978	0.91	0.7758
0.12	0.8135	0.32	0.7202	0.52	0.6800	0.72	0.7004	0.92	0.7806
0.13	0.8061	0.33	0.7169	0.53	0.6795	0.73	0.7031	0.93	0.7853
0.14	0.8000	0.34	0.7138	0.54	0.6792	0.74	0.7060	0.94	0.7898
0.15	0.7946	0.35	0.7107	0.55	0.6790	0.75	0.7090	0.95	0.7947
0.16	0.7894	0.36	0.7078	**0.56**	**0.6789**	0.76	0.7122	0.96	0.7995
0.17	0.7842	0.37	0.7050	0.57	0.6791	0.77	0.7156	0.97	0.8041
0.18	0.7792	0.38	0.7024	0.58	0.6794	0.78	0.7191	0.98	0.8040
0.19	0.7742	0.39	0.6998	0.59	0.6798	0.79	0.7228	0.99	0.7996
0.20	0.7694	0.40	0.6975	0.60	0.6804	0.80	0.7265	1.00	

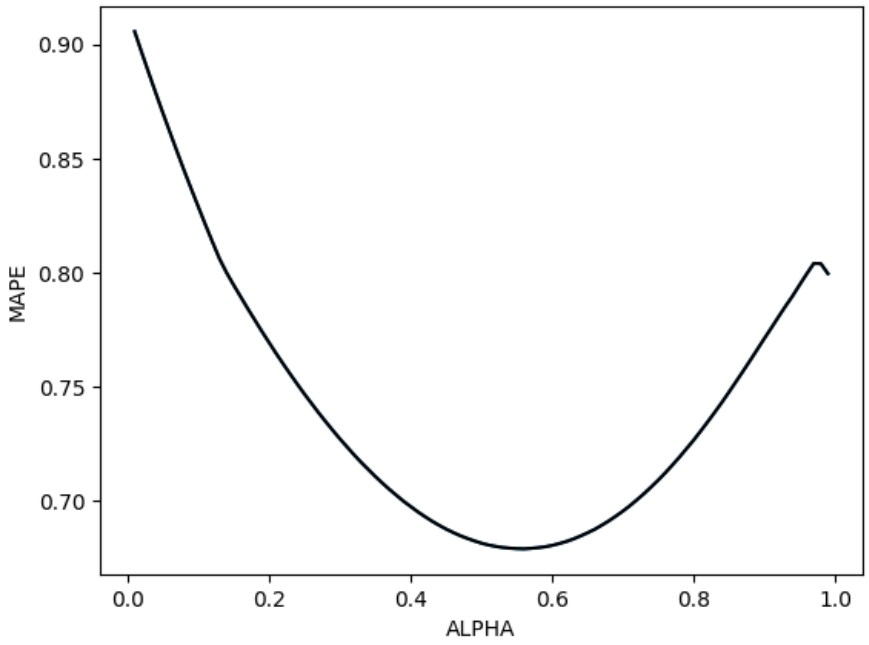


Figure 1. MAPEs of FEXGM(1, 1) model with α in (0,1).

Then a, b and c parameters can be obtained as

$$[a, b, c]^T = \left(B^T B\right)^{-1} B^T Y$$

and

$$\begin{aligned} a &= -0.0053 \\ b &= 29.9630 \\ c &= -33.8532. \end{aligned}$$

If we substitute the parameters into Eq. (3.12), we have

$$\hat{x}^{(\alpha)}(k) = -5577.28509 - 33.44487e^{-0.44k} + 5643.034798e^{-0.00537(1-k)}. \tag{4.2}$$

Then the restored values can be calculated using (4.2) as

$$\hat{X}^{(0.56)}(k) = (44.2100, 82.2752, 117.7748, 151.6814, 184.6214, 216.9987, 249.0737)$$

for k from 1 to 7. Then the restored values of the FEXGM(1,1) model are easily computed using CFD in (3.15) as

$$\hat{X}^{(0)}(k) = (44.2100, 51.6394, 57.5648, 62.4007, 66.8759, 71.2241, 75.5110).$$

The actual values and prediction results of FEXGM(1,1) are shown in Fig. 2.

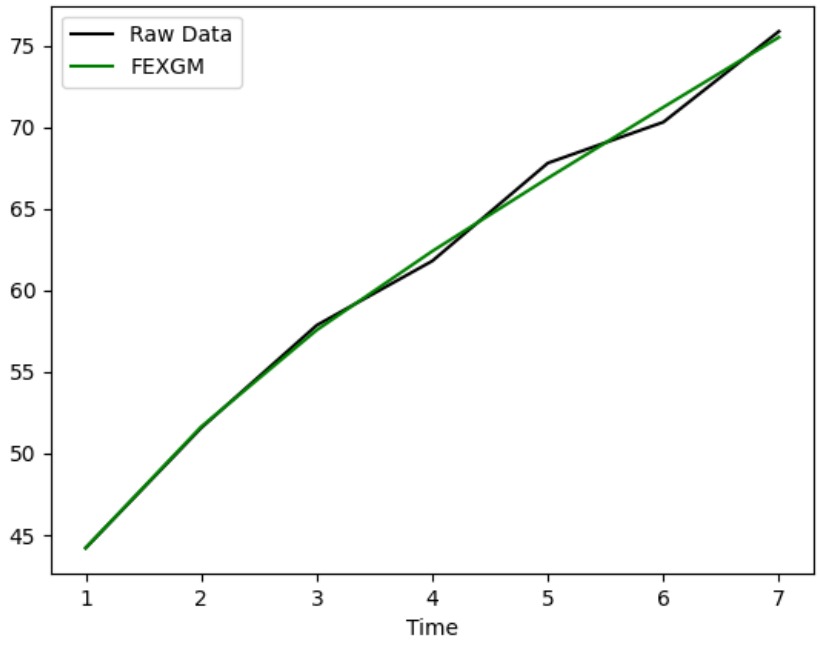


Figure 2. Forecasting and actual values of example of monotonically increasing sequence.

4.2. Example 2

As distinct from the first example, the raw data aren't a monotone increasing here. The raw data were selected as follows,

$$X^{(0)}(k) = (123.3, 132.0, 146.5, 161.6, 157.5, 152.3, 140.9, 138.6, 129.5).$$

In this example, we calculated MAPEs for each α value in the interval (0,1) with step 0.01. The results are plotted Fig. 3. By this way, the minimum MAPE is calculated at optimum α as $\alpha = 0.929$. The MAPE is computed for the optimum α as MAPE= 1.4298. It is clear that the prediction ability of the FEXGM(1,1) model is perfect.

4.2.1. Parameter estimation and conformable fractional accumulation. Computation of the CFA series of the original series is the first step to build the proposed model. For $\alpha = 0.929$, the CFA series can be calculated using Eq. (3.2) as,

$$X^{(0.929)}(k) = (123.3, 248.9610, 384.4681, 530.9201, 671.4127, 805.5196, 928.2379, 1047.8138, 1158.6084).$$

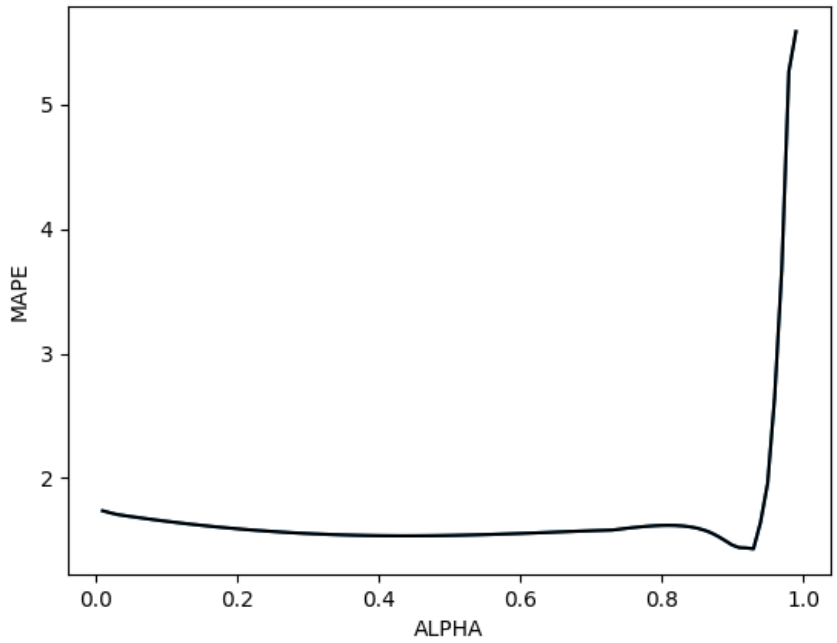


Figure 3. MAPEs of FEXGM(1, 1) model with α in (0,1).

The matrices B and Y can be constructed as

$$B = \begin{bmatrix} -186.1305 & 1 & -0.0638 \\ -316.7146 & 1 & -0.0594 \\ -457.6941 & 1 & -0.0553 \\ -601.1664 & 1 & -0.0515 \\ -738.4662 & 1 & -0.0480 \\ -866.8787 & 1 & -0.0447 \\ -988.0258 & 1 & -0.0416 \\ -1103.2111 & 1 & -0.0388 \end{bmatrix}, \quad Y = \begin{bmatrix} 125.6610 \\ 135.5070 \\ 146.4519 \\ 140.4926 \\ 134.1069 \\ 122.7182 \\ 119.5759 \\ 110.7945 \end{bmatrix}$$

Hence, we can calculate the parameters a, b, c using the least squares as

$$a = 0.6264$$
$$b = 1673.7260$$
$$c = 22447.0950.$$

By substituting the parameters into the response function Eq. (3.12) we have

$$\hat{x}^{(\alpha)}(k) = 2671.93483 - 2869.49173e^{-0.071k} + 124.18737e^{0.62641(1-k)}. \tag{4.3}$$

Then the restored values can be calculated using (4.3) as

$$\hat{X}^{(0.929)}(k) = (123.3000, 248.6818, 388.4175, 530.8415, 670.0600, 803.2409, 929.1669)$$

for k from 1 to 7. Then the restored values of the FEXGM(1,1) model are easily computed using CFD in (3.15) as

$$\hat{X}^{(0)}(k) = (123.3000, 131.7067, 151.0716, 157.1554, 156.0715, 151.2484, 144.5830, 137.1169, 129.4171).$$

The original raw series $X^{(0)}(k)$ and predicted values $\hat{X}^{(0)}(k)$ are plotted in Fig. 4.

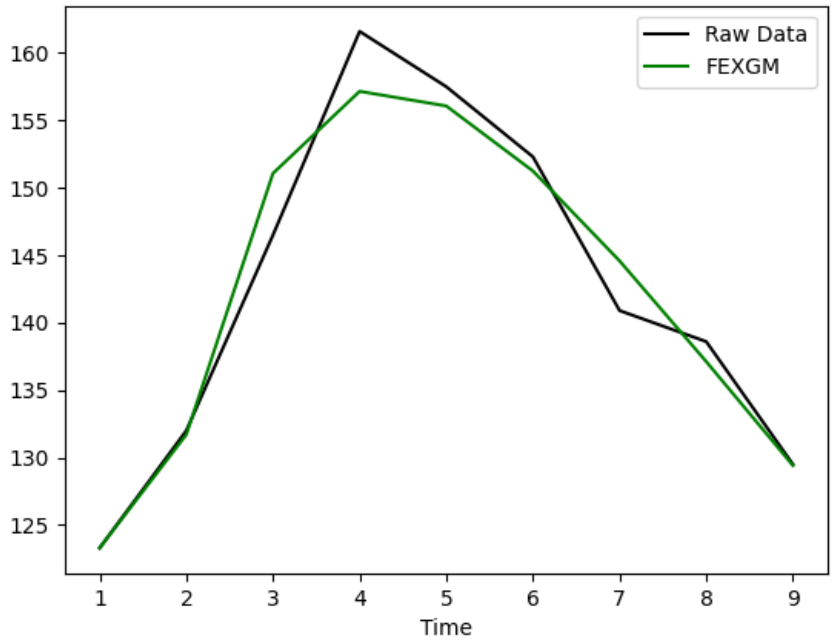


Figure 4. Forecasting and actual values of example of non-monotonically increasing sequence.

5. Real Case Applications for CO_2 Emissions

In this section, two real applications are given to validation of the FEXGM(1,1) model. The proposed model is applied to project carbon dioxide emissions in China and Brazil. China is one of the leading countries in the world both in terms of industry and population. Therefore, China's influence on greenhouse gas emission values around the world is curious. The CO_2 emission values in Brazil, which is in a different position from China in terms of industry, geographical location and population density, were also considered as a good example for comparison.

5.1. Collection of data

The CO_2 emission data of China and Brazil were collected from the Statistical Review of World Energy-BP [2]. The collected data are shown in Table 2 and Table 5, as raw data. Raw data were reserved for model fitting and the forecasting.

5.2. Prediction of CO_2 emission in China based on FEXGM(1,1)

In this section, we use the proposed grey model to predict CO_2 emissions in China. CO_2 emission data in China from 2010 to 2020 is used to test for the efficacy and practicality of the novel fractional FEXGM(1,1) model. In this application, we split the raw data into two groups. The first of these was used to build the models, and the other group was used to test the prediction capabilities of the models. The first group, which contains the CO_2 emissions from 2010 to 2018, is utilized to construct each of the six effective grey models independently. The second group, which consists of CO_2 emissions from 2019 to 2020, is utilized to assess how well these grey models predicted the future. Afterward, using the computational steps described in Section 3.5, we list all the values in the range $(0, 1)$ with a step size of 0.001 and then we choose the α that minimizes the $MAPE_{fit}$ at the optimum value. The Brute Force algorithm yields the optimal α value of $\alpha = 0.787$. Figure 5 shows that as the α deviates from its optimal value, the values of MAPE rise. The optimum parameters are calculated as $\alpha = 0.787$, $a = -0.011683$, $b = 4709.75726$ and $c = -17838.65350$.

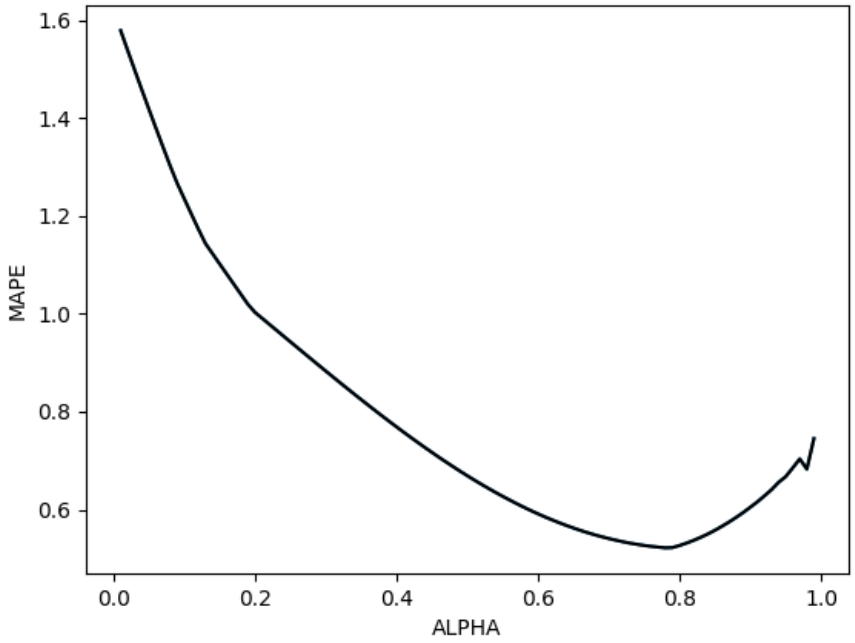


Figure 5. MAPEs of FEXGM(1, 1) model for CO_2 emission in China with α in (0,1).

According to the Eq. (3.12), we can obtain the response function as

$$\hat{x}^{(0.787)}(k) = -403129.0987 - 16911.08449e^{-0.213k} + 424941.6954e^{0.011683(1-k)}. \quad (5.1)$$

Furthermore, the FEXGM(1,1) model is compared with the five effective models, including the GM(1,1), EXGM(1,1), CCFGM(1,1), CFGM(1,1) and PFAGM(1,1). In Table 2, mean absolute percentage errors, and prediction outcomes of the most current models are displayed. The following important conclusions can be reached from Table 2:

i. The $MAPE_{fit}$ values for the six grey models are 0.7561%, 0.6079%, 0.6401%, 0.6503%, **0.3784**% and 0.5208%, as shown in Table 2. The PFAGM(1,1) model currently has the least $MAPE_{fit}$ value, followed by the FEXGM(1,1) model.

ii. The $MAPE_{pre}$ values of seven models are calculated as 1.0443%, 1.8342%, 2.5300%, 2.3916%, 12.1560%, and **0.4537**%, respectively. Therefore, the FEXGM(1,1) model has the best fitting performance across the fitting period (see Table 3).
iii. Table 2 shows that for the total MAPEs of seven models are, in order of, 0.8137%, 0.8532%, 1.0181%, 0.9986%, 2.7339%, and **0.5074**%, for the entire period. Hence, across the entire time period, the FEXGM(1,1) model's performance is the best (see Fig. 6).
iv. With the suggested FEXGM(1,1) model, China's CO_2 emissions can be accurately estimated. So that, Table 4 provides the forecasted values for China's CO_2 emissions.
v. Simulative curves with six models for CO_2 emissions of Brazil are given in Fig. 7. Also, the actual values and forecasting values of CO_2 emissions of Brazil are given in Fig. 8.

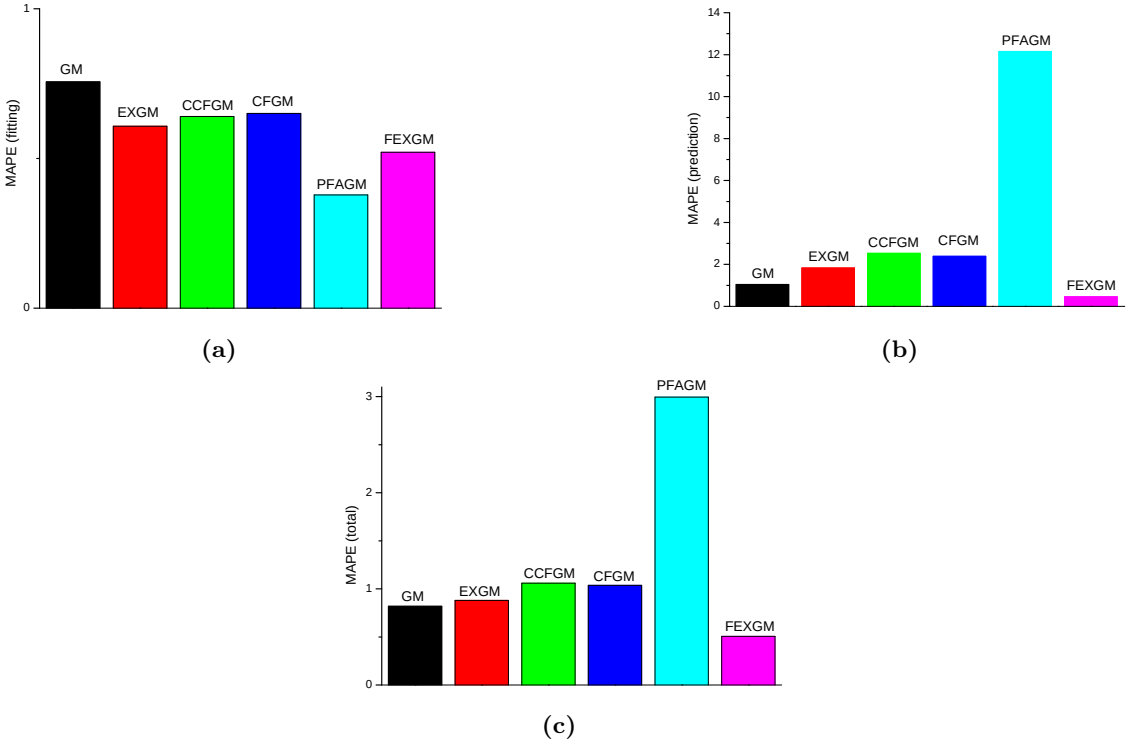


Figure 6. MAPE values of the models for CO_2 emission in China (a)fitting, (b) prediction, (c) total.

5.3. Prediction of CO_2 emission in Brazil based on FEXGM(1,1)

We use the novel FEXGM(1,1) model to predict CO_2 emissions in Brazil in this section. To evaluate the usefulness and applicability of the suggested grey model, the raw data from [2] is used. The suggested model was also contrasted with the five efficient models, including the GM(1,1), EXGM(1,1), CCFGM(1,1), CFGM(1,1,k) and PFAGM(1,1).

Additionally, the prediction accuracy of these grey models is evaluated using the mean absolute percentage error (MAPE). nitially, the raw data was divided into two groups. The first group, which consists of CO_2 emissions from 2010 to 2018, is used to create individual models for each of six grey models. To assess the efficacy of these grey models' predictions, the second group, which consists of CO_2 emissions from 2019 to 2020, is employed. We count all the values in the range $(0, 1)$ width a step of 0.001 in this section,

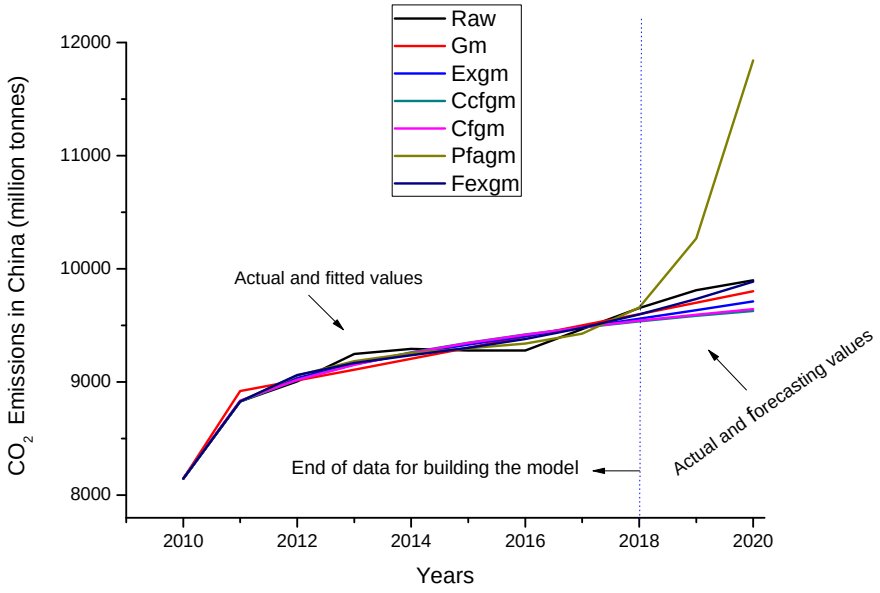


Figure 7. Simulative curves with six models for CO_2 emissions of China.

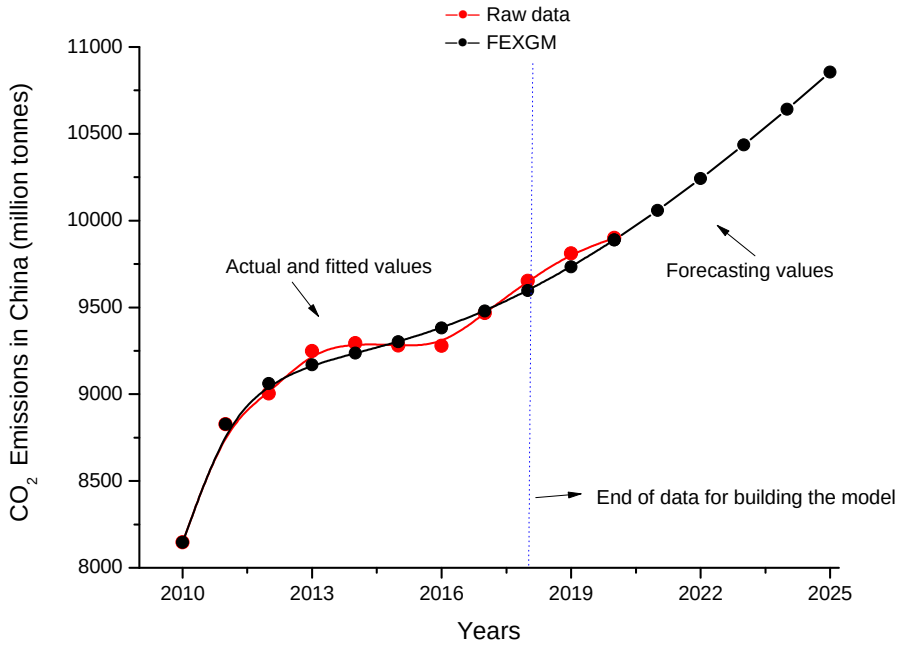


Figure 8. Actual values and forecasting values of CO_2 emissions of China.

then we employ the computational steps described in Section 3.5 to choose the α that minimizes the $MAPE_{fit}$ as the optimum value. Optimum α is evaluated as $\alpha = 0.958$ by using the Brute Force algorithm. In addition, all the values of α in $(0, 1)$ and their MAPE values are given in Fig. 9. Therefore, the model parameters are obtaned as $a = 0.51865$, $b = 6966.76022$ and $c = 158304.85660$ for $\alpha = 0.958$. If we substitute the parameters into Eq. (3.12), we have

$$\hat{x}^{(0.958)}(k) = 13432.47696 - 13949.01437e^{-0.042k} + 345.91139e^{0.51865(1-k)}. \tag{5.2}$$

Table 2. Fitting values and prediction values of comparative five model with proposed model for CO_2 emissions in China (million tonnes).

Year	Raw Data	GM	EXGM	CCFGM	CFGM	PFAGM	FEXGM
2010	8145.8	8145.8000	8145.8000	8145.8000	8145.8000	8145.8000	8145.8000
2011	8827.2	8919.8407	8828.1589	8827.3607	8835.4504	8827.1132	8827.2104
2012	9004.2	9013.9206	9035.2045	9019.3513	9018.6533	9057.0302	9062.0097
2013	9247.4	9108.9928	9157.6055	9156.0118	9151.6008	9183.6040	9168.3089
2014	9293.2	9205.0678	9249.2369	9261.8086	9256.5896	9254.9775	9236.3974
2015	9279.7	9302.1561	9329.9210	9347.8165	9343.6919	9298.4737	9302.3121
2016	9279.0	9400.2684	9406.9531	9420.0165	9418.3383	9341.0912	9381.1463
2017	9466.4	9499.4155	9483.0200	9482.0148	9483.7986	9428.0767	9478.6691
2018	9652.7	9599.6083	9559.1130	9536.1577	9542.1972	9658.5883	9596.2346
$MAPE_{fit}$		0.7561	0.6079	0.6401	0.6503	0.3784	**0.5208**
2019	9810.5	9700.8579	9635.6000	9584.0594	9594.9938	10269.7628	9733.0974
2020	9899.3	9803.1754	9712.6194	9626.8788	9643.2354	11842.6113	9887.5750
$MAPE_{pre}$		1.0443	1.8342	2.5300	2.3916	12.1560	**0.4537**
$MAPE_{tot}$		0.8137	0.8532	1.0181	0.9986	2.7339	**0.5074**

Table 3. MAPE values of prediction values for CO_2 emissions in China.

Year	GM	EXGM	CCFGM	CFGM	PFAGM	FEXGM
2019	1.1175	1.7827	2.3081	2.1966	4.6813	0.7889
2020	0.9710	1.8857	2.7519	2.5866	19.6307	0.1184

Table 4. Forecasted CO_2 emissions of China.

Year	Forecasting values
2019	9733.0544
2020	9887.5248
2021	10057.5924
2022	10241.2161
2023	10436.4844
2024	10641.6887
2025	10855.3424

In addition, the FEXGM model compared with the five effective models, including the GM(1,1), EXGM(1,1), CCFGM(1,1), CFGM(1,1) and PFAGM(1,1). The prediction values and the MAPEs of the models are given in Table 5. According to the Table 5, the principal results can be drawn as follows:

i. Table 5 exposes that, the six grey models' $MAPE_{fit}$ values are 4.9186%,2.4924%, **1.8969**%, 2.0110%, 2.0580% and 1.9999%, respectively. The CCFGM(1,1) model has the lowest $MAPE_{fit}$ value while the FEXGM(1,1) model has the second lowest $MAPE_{fit}$ in here.

ii. The $MAPE_{pre}$ values of seven models are calculated as 9.0837%, 2.1580%, 3.0705%, 2.8285%, 3.4699%, and **1.8478**%, respectively. So that, in the fitting period, the fitting performance of the FEXGM(1,1) model is best (see Table 6).

iii. It can be seen from Table 5 that for the whole period, the total MAPEs of six models are 5.7516%, 2.4255%, 2.1316%, 2.1745%, 2.3403%, and **1.9695**%, respectively. It is seen that the FEXGM(1,1) model obtains better results than the other discrete grey models (see Fig. 10).

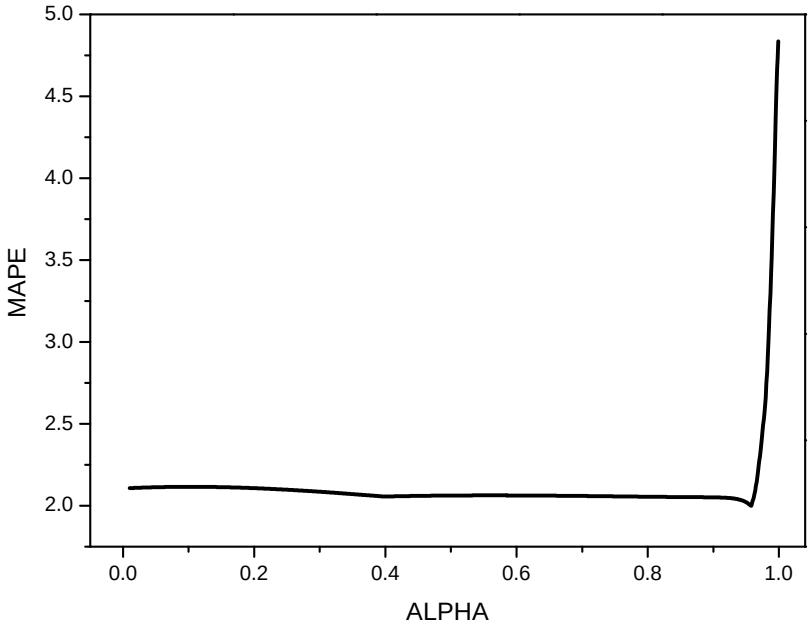


Figure 9. MAPEs of FEXGM(1, 1) model for CO_2 emission in Brazil with α in (0,1).

Table 5. Fitting values and prediction values of comparative five model with proposed model for CO_2 emissions in Brazil (million tonnes).

Year	Raw Data	GM	EXGM	CCFGM	CFGM	PFAGM	FEXGM
2010	403.1	403.1000	403.1000	403.1000	403.1000	403.1000	403.1000
2011	427.2	461.1000	418.6935	427.0956	423.2513	418.3009	422.1934
2012	445.7	462.1241	472.7672	465.4177	462.7722	465.6330	465.0973
2013	485.8	463.1505	486.0446	482.7810	480.9951	485.7938	483.5180
2014	508.0	464.1791	484.4510	487.3016	486.2523	489.5432	487.3056
2015	490.7	465.2100	477.5213	483.4346	483.0136	483.2497	482.4391
2016	454.1	466.2432	468.7601	473.9250	474.0195	471.3154	472.5432
2017	460.6	467.2787	459.4541	460.5897	461.0865	457.2633	459.8121
2018	445.5	468.3165	450.0739	444.6882	445.4795	444.4289	445.5734
$MAPE_{fit}$		4.9186	2.4924	1.8969	2.0110	2.0580	**1.9999**
2019	444.9	469.3566	440.7901	427.1211	428.1078	436.5827	430.6289
2020	417.5	470.3990	431.6627	408.5450	409.6394	438.6691	415.4624
$MAPE_{pre}$		9.0837	2.1580	3.0705	2.8285	3.4699	**1.8478**
$MAPE_{tot}$		5.7516	2.4255	2.1316	2.1745	2.3403	**1.9695**

iv. The CO_2 emission of Brazil can be predicted reliably with the proposed FEXGM (1,1) model. Hence, the forecasting results of CO_2 emission of Brazil are indicated in Table 7.

v. Simulative curves with six models for CO_2 emissions of Brazil are given in Fig. 11. Also, the actual values and forecasting values of CO_2 emissions of Brazil are given in Fig. 12.

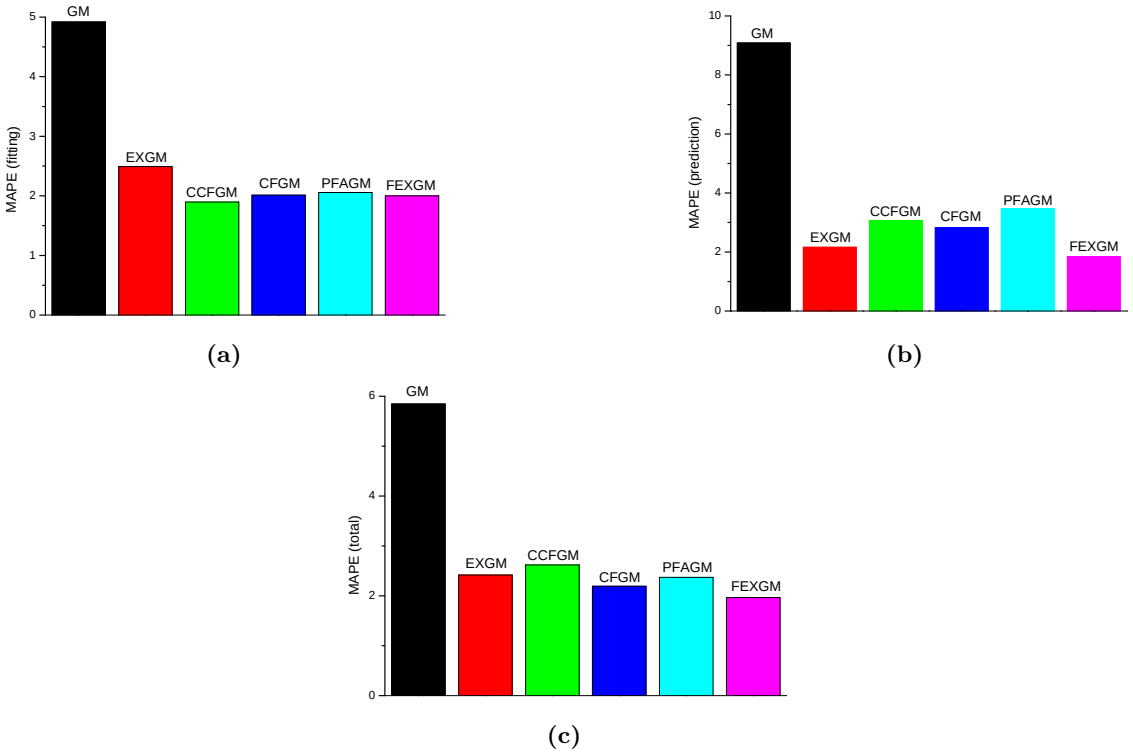


Figure 10. MAPE values of the models for CO_2 emission in Brazil (a)fitting, (b) prediction, (c) total.

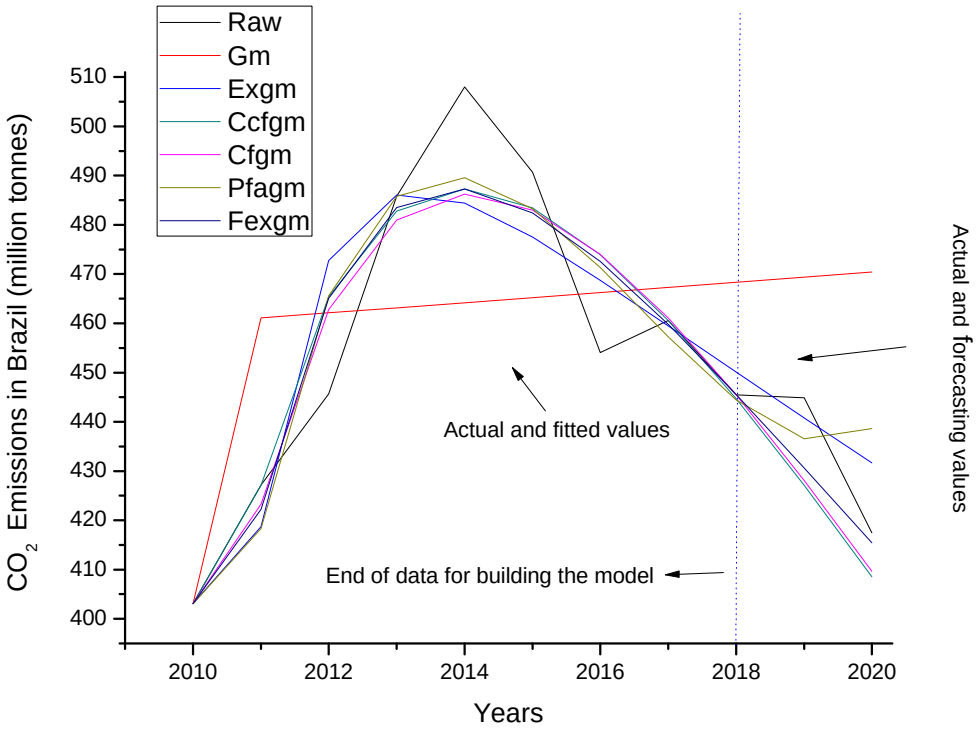


Figure 11. Simulative curves with six models for CO_2 emissions of Brazil.

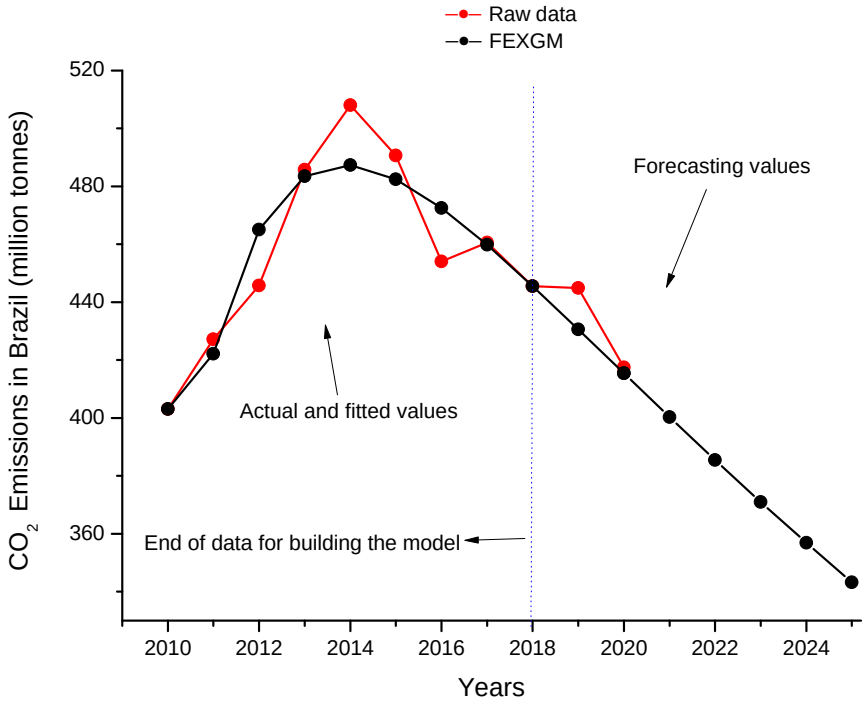


Figure 12. Actual values and forecasting values of CO_2 emissions of Brazil.

Table 6. MAPE values of prediction values for CO_2 emissions in Brazil.

Year	GM	EXGM	CCFGM	CFGM	PFAGM	FEXGM
2019	5.4971	0.9237	3.9961	3.7743	1.8694	3.2076
2020	12.6704	3.3922	2.1448	1.8827	5.0704	0.4880

Table 7. Forecasting values of CO_2 emissions of Brazil using FEXGM.

Year	Forecasting values (million tonnes)
2019	430.6204
2020	415.4530
2021	400.3544
2022	385.4974
2023	370.9834
2024	356.8700
2025	343.1881

6. Conclusion

It is clear that, the reduction of CO_2 emissions will be a positive situation for human health and the environment. Exposures to air toxics can also cause noncancerous health effects, such as neurological, cardiovascular, respiratory, reproductive and immune system damage. Today, many countries have enroled the Paris Agreement on climate change, which calls for keeping the global temperature to 1.5řC above pre-industrial era levels. If we continue to pump out the emissions that cause climate change, however, temperatures will continue to rise well beyond 1.5řC, to levels that threaten the lives and livelihoods of people everywhere [18].

Adaptation measures to be taken for carbon emissions will save people's lives as well as cost savings. It is clear that the effects of climate change have different severity and structure in different parts of the world. In this case, the measures to be taken are carried out at regional or local scales.

Since fractional calculations are important for mathematical modeling, there are many scientists who have proposed new ideas or operators on the fractional grey model. Accordingly, we have developed a novel optimization for the FAGM (1,1) and EXGM(1,1) models in this study. By developing a different perspective, in order to increase the positive effect of fractional order on the parameter estimation, more active role in the whitening equation was given to the α parameter. The numerical results show that the proposed fractional grey forecasting model aims to achieve very effective performance. The fundamental parameters (a, b and c) of the model can be dynamically arranged according to the real world systems. A Brute-Force algorithm was used to determine the optimal value of the fractional order, α. Comparison results indicate, FEXGM(1,1) performs better than those achieved by the other grey models such as GM(1,1), EXGM(1,1), CCFGM(1,1), CFGM(1,1,k) and PFAGM(1,1). However, they can all be employed for estimations.

Using two illustrative examples and two real case studies in Sections 4 and 5, we show that the percentage error values of the proposed fractional grey model are very low. We estimated the CO_2 emission values of China and Brazil using the FEXGM(1,1) model for the next seven years. According to the forecast results, China's CO_2 emission value will gradually increase and Brazil's CO_2 emission value will gradually decrease. As a result, it can be suggested that necessary measures should be implemented or strengthened, particularly in countries like China where both population and industrial density are high. The forecast results in this article can be a guide for the planning of the relevant departments. Furthermore, it can be seen that the proposed model effectively predicts data for monotonic or non-monotonic data sets.

The proposed FEXGM(1,1) model may play an significant role in enriching the theoretical mechanism of grey forecasting theory. In addition, the novel FEXGM(1,1) model can also be used for other applications.

Credit author statement

Halis Bilgil: Conceptualization, Methodology, Data Collection, Formal Analysis and Interpretation of Results, Writing original draft, Writing review & editing.
Ümmügülsüm Erdinç: Conceptualization, Methodology, Supervision, Formal Analysis and Interpretation of Results, Writing review & editing.

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